La matematica dei gusci

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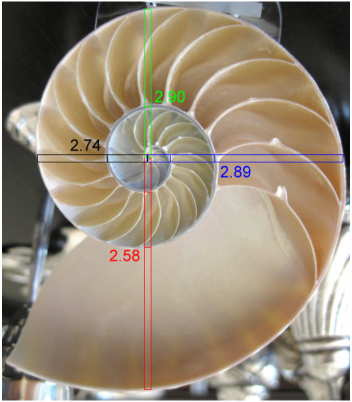
Modello di Raup

This work focuses on the process of growth of shells. We were mainly interested to study the Model of Raup. Our first aim was, in fact, to understand the meaning of the parameters of moodel's Raup, to reproduce the model in Mathematica language and use it to model some conch shells taken from the nature.



D'Arcy Thompson

Conch shells of almost all molluscs are characterized by a common symmetry that is based on a tube that turns in a spiral about an axis. D'Arcy Thompson made the following observations:

- growth by addition: conch shells increase their dimensions by adding new material to what already exists.
- invariance of form: in their growth conch shells always conserve the same shape and contain in themselves all the previous phases of growth.
- universal form: If we take two conches of the same species but of different ages we can see that the one shape is the enlargement of the second

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History of Raup's Model

- source of the model: the discovery that shells such as Nautilus can be modelled by logarithmic spirals dates back at least to the eighteenth century
- validity of the model: With D'Arcy Thompson, measurement and analysis were used subsequently to demonstrate logarithmic spirals in a variety of molluscs
- Raup's contribute: Raup(1966) demonstrates that many molluscan shells are helicoid logarithmic spiral cones.

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Modello di Raup

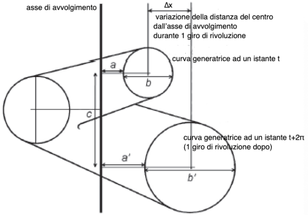
Raup model of ammonoids.

In 1967 Dave Raup was interested in the shell form of ammonoids, an extinct group of cephalopod molluscs, related to the chambered nautilus that still survives in today's oceans. Raup decided that the essence of the form could be given by a two-dimensional logarithmic spiral model. In polar coordinates, it has three geometric parameters: radius r, coiling angle ϕ, and tangent angle α and is described by the equation

$$r(\phi) = r_0 e^{cot\alpha \ \phi}$$

r_0 is the initial radius $r(\phi)$ is the radius at coiling angle ϕ. The variables $r(\phi)$ and ϕ change while the spiral grows. The tangent angle α is a constant parameter. Invariance of tangent angle means that the form of logarithmic spiral is invariant in the growth; that is, the proportions of the spiral remain the same regardless of the size of the spiral.

Parameters of Raup Model



- Diameter of umbilicus D: a proportion of total diameter

$$D = \frac{a}{a+b}$$

- Whorl translation T: distance centre of whorl translates parallel to coiling axis in one rotation, divided by increase in distance of centre of whorl from coiling axis

$$T = \frac{c}{x}$$

- Whorl expansion rate W: ratio of whorl diameters separated by one rotation about the coiling axis

$$W = \frac{b'}{b}$$

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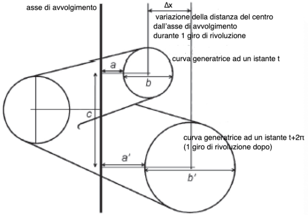

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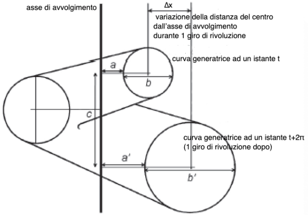

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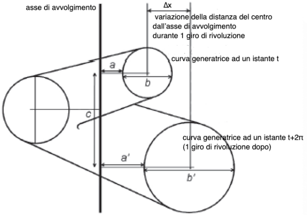

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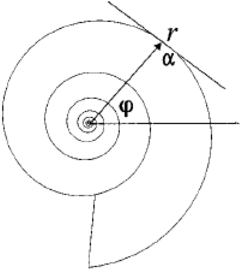
Parameter W: whorl expansion rate

Raup preferred to measure ratios of radii rather than tangent angles (it is easier!), thus he designed a new parameter to replace the tangent angle, which he named the 'whorl expansion rate', or W , given by

$$W = \frac{r(\phi)}{r_0}^{\frac{2\pi}{\phi}}$$

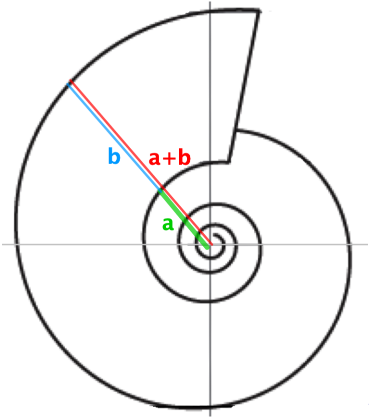
He then re-wrote equation (4.1) in the following fashion, in terms of his new parameter W as

$$r(\phi) = r_0 W^{\frac{\phi}{2\pi}}$$

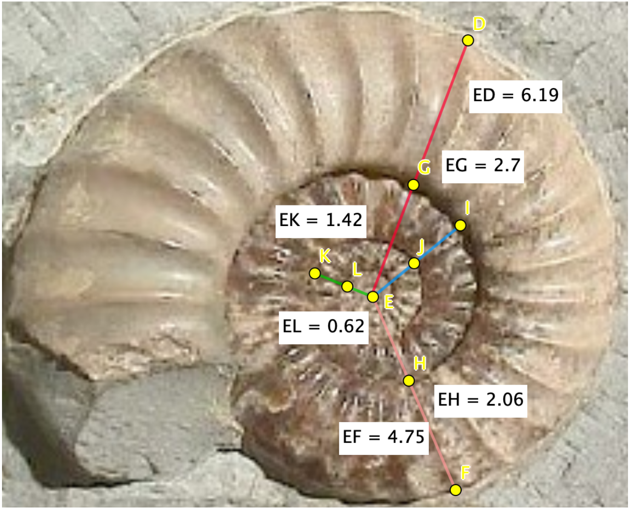


Parameter D

$D = \frac{a}{a+b}$ is the ratio between the distance a of the inner margin and the distance $a + b$ of the outer margin of the generating curve from the coiling axis.
It remains unchanged in the growth process.



Parameter D in the ammonoid



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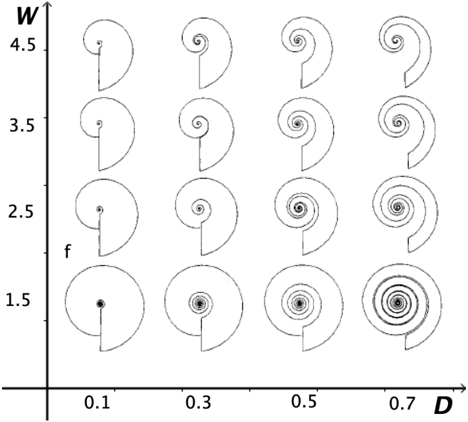
Modello di Raup

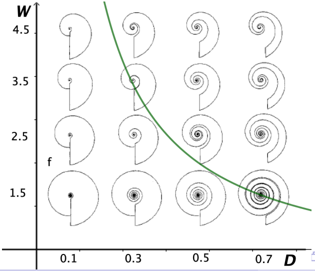
Morphospace of ammonoids

Raup obtained different spirals by changing values of parameters W and D and composed the morphospace of ammonoid form:



Snugness condition

The hyperbola $W = \frac{1}{D}$ is the whorl-overlap boundary that delimits the shells that have whorls that overlap one another from the region shells where the whorls do not touch one another. Ammonoids that lie on the hyperbola have shells that have whorls that exactly touch one another, neither overlapping nor separating. Very few ammonites satisfy this condition. Tha majority of ammonoids in nature occur to the region where $W = \frac{1}{D}$ condition holds.



Spirale logaritmica

In generale, il modello di Raup è una superficie:

$$F(t, \phi) = (x(t, \phi), y(t, \phi), z(t, \phi))$$

ma alcuni tipi di conchiglie come le ammoniti o il nautilus sono caratterizzate dall'assenza di scorrimento lungo l'asse (il parametro T è nullo). Bastano i due parametri D, W. In tali conchiglie la coordinata z rimane costante con valore 0 e di conseguenza anche la coordinata $\phi = 0$ rimane costante durante il processo di crescita. Possiamo descrivere la conchiglia con la curva

$$f(t) = F[t, 0] = (x(t), y(t), 0)$$

che è nota come *spirale logaritmica*:

$$x(t) = A\, W^{\frac{t}{2\pi}} cos\ t, \quad y(t) = A\, W^{\frac{t}{2\pi}} sin\ t$$

Spirale logaritmica: ammonite

Vogliamo fare il modello di un'ammonite usando la *spirale logaritmica*:

$$x(t) = A\, W^{\frac{t}{2\pi}} \cos t, \quad y(t) = A\, W^{\frac{t}{2\pi}} \sin t$$

Determiniamo il parametro cartteristico W (whorl) dall'osservazione della foto di un'ammonite

Spirale logaritmica: ammonite

La *spirale logaritmica*:

$$x(t) = A\,W^{\frac{t}{2\pi}} \cos t, \quad y(t) = A\,W^{\frac{t}{2\pi}} \sin t$$

è un modello della nostra ammonite con un valore di W prossimo a 2.6